

Exercise 53

Suppose that $|f(x)| \leq g(x)$ for all x , where $\lim_{x \rightarrow a} g(x) = 0$. Find $\lim_{x \rightarrow a} f(x)$.

Solution

Suppose that

$$|f(x)| \leq g(x).$$

Rewrite the inequality.

$$-g(x) \leq f(x) \leq g(x)$$

Take the limit of all sides as $x \rightarrow a$.

$$\lim_{x \rightarrow a} [-g(x)] \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

Bring the minus sign in front.

$$-\lim_{x \rightarrow a} g(x) \leq \lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$$

Evaluate the limits on the left and right sides using the fact that $\lim_{x \rightarrow a} g(x) = 0$.

$$-(0) \leq \lim_{x \rightarrow a} f(x) \leq 0$$

$$0 \leq \lim_{x \rightarrow a} f(x) \leq 0$$

Therefore, by the Squeeze Theorem,

$$\lim_{x \rightarrow a} f(x) = 0.$$